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A Detailed Survey of Cutting-Edge Strategies for Solving Initial Value Problems

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ABSTRACT—

The numerical solution of initial value problems (IVPs) is essential in the study of differential equations, with applications across engineering, physics, biology, and finance. Among the various numerical methods, the Modified Euler's method and Taylor's method stand out for their unique approaches and effectiveness. The Modified Euler's method, a second-order Runge-Kutta technique, enhances the basic Euler's method by incorporating an extra derivative evaluation, striking a balance between computational efficiency and accuracy. In contrast, Taylor's method uses Taylor series expansion for approximation, achieving high precision by considering higher-order derivatives. While adding more terms from the Taylor series increases accuracy, it also raises computational complexity. Research indicates that the Modified Euler's method is simpler and less resource-intensive, while Taylor's method excels in accuracy when higher-order derivatives are readily available. Comparative studies reveal that both methods have distinct advantages and drawbacks, with the choice often depending on specific problem requirements, including desired accuracy and computational resources. Recent developments have also introduced adaptive step size techniques and hybrid methods, broadening the applicability of both approaches to a variety of IVPs.

KEY WORDS:— initial value problems; Modified Euler's method; Runge-Kutta method; Taylor's method; Legendre Neural Network

I. Background:

State-of-the-art techniques for solving initial value problems (IVPs) are essential in differential equations and have broad applications in engineering, physics, biology, and finance. Among the numerous numerical methods available, the Modified Euler's method and Taylor's method are particularly notable for their unique approaches and effectiveness. The Modified Euler's method, a second-order Runge-Kutta technique, enhances the basic Euler's method by adding an extra derivative evaluation, which improves accuracy while maintaining computational efficiency. In contrast, Taylor's method employs Taylor series expansion for approximation, achieving high precision by incorporating higher-order derivatives. While this method's accuracy increases with additional terms from the Taylor series, it also introduces greater computational complexity. Recent advancements have seen these methods integrated with adaptive step size techniques and hybrid approaches, further expanding their applicability to a broader range of IVPs.

2. Review of Different Methods:

2.1 Initial Value Problems (IVPs)

Anitescu et al. [1] proposed a method for solving partial differential equations using artificial neural networks and an adaptive collocation strategy. In this procedure, a coarse grid of training points is used at the initial training stages, while more points are added at later stages based on the value of the residual at a larger set of evaluation points. This method increases the robustness of the neural network approximation and can result in significant computational savings, particularly when the solution is non-smooth. Numerical results are presented for benchmark problems for scalar-valued PDEs, namely Poisson and Helmholtz

equations, as well as for an inverse acoustics problem.

Abdeljawad et al. [2] proposed fractional operators with nonsingular Mittag-Leffler kernels, a study initiated recently by Atangana and Baleanu, from order $\alpha \in [0, 1]$ to higher arbitrary order and we formulate their correspondent integral operators. We prove existence and uniqueness theorems for the Caputo (ABC) and Riemann (ABR) type initial value problems by using the Banach contraction theorem. Then we prove a Lyapunov type inequality for the Riemann type fractional boundary value problems of order $2 < \alpha \leq 3$ in the frame of Mittag-Leffler kernels. Illustrative examples are analyzed and an application as regards the Sturm-Liouville eigenvalue problem in the sense of this fractional calculus is given as well.

Cano et al. [3] proposed Lawson methods suffer from a severe order reduction when integrating initial boundary value problems where the solutions are not periodic in space or do not satisfy enough conditions of annihilation on the boundary. However, in a previous paper, a modification of Lawson quadrature rules has been suggested so that no order reduction turns up when integrating linear problems subject to time-dependent boundary conditions. In this paper, we describe and thoroughly analyse a technique to avoid also order reduction when integrating nonlinear problems. This is very useful because, given any Runge-Kutta method of any classical order, a Lawson method can be constructed associated to it for which the order is conserved.

Chen et al. [4] proposed specifying a discrete sequence of hidden layers, we parameterize the derivative of the hidden state using a neural network. The output of the network is computed using a blackbox differential equation solver. These continuous-depth models have constant memory cost, adapt their evaluation strategy to each input, and can explicitly trade numerical precision for speed. We demonstrate these properties in continuous-depth residual networks and continuous-time latent variable models. We also construct continuous normalizing flows, a generative model that can train by maximum likelihood, without partitioning or ordering the data dimensions. For training, we show how to scalably backpropagate through any ODE solver, without access to its internal operations. This allows end-to-end training of ODEs within larger models.

Gu et al. [5] proposed to utilize the radial basis function (RBF) interpolation to modify several finite difference methods and thus enhance the performance in terms of local convergence. In this work, we choose multiquadric RBFs as the interpolation basis and find the conditions of the shape parameter that could enhance accuracy. The rate of convergence of each modified method is at least the same as the original one and can be further improved by making the local truncation error vanish. In that sense, the proposed adaptive method is optimal. Compared to the linear multistep methods, the proposed adaptive RBF multistep methods exhibit higher order convergence. We provide the analysis of consistency and stability with numerical results that support our claims.

Prakash et al. [6] proposed how we can extend the invariant subspace method to two-dimensional time-fractional non-linear PDEs. More precisely, the systematic study has been provided for constructing the various dimensions of the invariant subspaces for the two-dimensional time-fractional generalized convection-reaction diffusion-wave equation along with the initial conditions for the first time. Additionally, the special types of the above-mentioned equation are discussed through this method separately such as reaction-diffusion-wave equation, convection-diffusion wave equation and diffusion-wave equation. Moreover, we explain how to derive variety of exact solutions for the underlying equation along with initial conditions using the obtained invariant subspaces. Finally, we extend this method to two dimensional time-fractional non-linear PDEs with time delay. Also, the effectiveness and applicability of the method have been illustrated through the two-dimensional time-fractional cubic non-linear convection-reaction-diffusion-wave equation with time delay. In addition, we observe that the obtained exact solutions can be viewed as the combinations of Mittag-Leffler function and polynomial, exponential and trigonometric type functions.

Mall et al. [7] proposed a new method based on single layer Legendre Neural Network (LeNN) model has been developed to solve initial and boundary value problems. In the proposed approach a Legendre polynomial based Functional Link Artificial Neural Network (FLANN) is developed. Nonlinear singular initial value problem (IVP), boundary value problem (BVP) and system of coupled ordinary differential equations are solved by the proposed approach to show the reliability of the method. The

hidden layer is eliminated by expanding the input pattern using Legendre polynomials. Error back propagation algorithm is used for updating the network parameters (weights). Results obtained are compared with the existing methods and are found to be in good agreement. Below Table 1.1 shows initial value problems analysis.

Table 2.1: Initial Value Problems Analysis

Author name and Reference	Techniques used	Merits	Demerits
Anitescu et al. [1]	Artificial Neural Networks (ANNs)	Enhances robustness, reduces computational costs, handles non-smooth solutions effectively, and adapts flexibly to various PDE problems.	Adds complexity in implementation, potential increased overhead, computationally intensive residual calculations, and possible training instability.
Abdeljawad et al. [2]	fractional operators with nonsingular Mittag-Leffler kernels, formulated integral operators	Extends fractional calculus to higher orders, provides theoretical results for initial and boundary value problems, and includes applications to Sturm-Liouville eigenvalue problems.	Complex theoretical framework, potentially challenging implementation, and higher-order fractional operators may complicate the analysis and computation.
Cano et al. [3]	Modified Lawson quadrature rules to avoid order reduction in nonlinear problems	Prevents order reduction in nonlinear problem integration, and allows for order preservation with any classical Runge-Kutta method.	Potentially complex implementation, and effectiveness depends on the specific modification and problem characteristics.
Chen et al. [4]	Parameterized hidden state derivatives with a neural network	Constant memory cost, adaptable evaluation strategy, trade-off between precision and speed, and end-to-end training of ODEs within larger models.	Complexity in implementation and training, potential challenges with scalability and numerical stability, and reliance on effective backpropagation through ODE solvers.
Gu et al. [5]	Utilized radial basis function (RBF) interpolation with multiquadric RBFs to modify finite difference methods	Enhances local convergence, maintains or improves convergence rate compared to original methods, and exhibits higher order convergence than linear multistep methods.	Complexity in choosing optimal shape parameters, potential computational overhead with RBF interpolation, and dependency on accurate numerical analysis for consistency and stability.
Prakash et al. [6]	derived exact solutions, and extended to nonlinear PDEs with time delay	Provides systematic study and exact solutions for complex fractional PDEs, extends to nonlinear cases, and combines Mittag-Leffler functions with polynomial, exponential, and trigonometric functions.	Complexity in constructing invariant subspaces, potential challenges in handling nonlinear terms with time delay, and dependence on intricate analysis for exact solution derivation.
Mall et al. [7]	Legendre polynomial-based Functional Link Artificial Neural Network (FLANN)	Simplifies network architecture by eliminating the hidden layer, achieves good agreement with existing methods, and effectively solves nonlinear and coupled ODE problems.	Limited to single-layer architecture, may face challenges in handling very complex problems compared to multi-layer networks, and effectiveness depends on the choice of Legendre polynomials.

2.2 Modified Euler's Method

Modified Euler's Method, also known as Heun's method, is a numerical technique used for solving ordinary differential equations (ODEs) with improved accuracy compared to the basic Euler's method. It involves a predictor-corrector approach where an initial estimate of the solution is refined using an average of the slope at the beginning and end of the interval. This method offers a second-order accuracy, providing a balance between computational efficiency and precision, making it suitable for various engineering and scientific problems.

Mao et al. [8] proposed several numerical methods have been developed to study the strong convergence of the numerical solutions to stochastic differential equations (SDEs) under the local Lipschitz condition. These numerical methods include the tamed Euler–Maruyama (EM) method, the tamed Milstein method, the stopped EM, the backward EM, the backward forward EM, etc. In this paper we will develop a new explicit method, called the truncated EM method, for the nonlinear SDE $dx(t) = f(x(t))dt + g(x(t))dB(t)$ and establish the strong convergence theory under the local Lipschitz condition plus the Khasminskii-type condition $x\ T\ f(x) + p-1/\ 2\ |\ g(x)|\ ^{2} \leq K(1+\ |\ x\ |\ ^{2}\ )$. The type of convergence specifically addressed in this paper is strong-L q convergence for $2 \leq q < p$, and p is a parameter in the Khasminskii-type condition.

Alobaid et al. [9] proposed an additional particle–grid is applied, in which the physical values of solid phase are computed. To investigate the influence of the particle–grid application on the simulation accuracy, the numerical results obtained by Euler–Lagrange approach combined with a deterministic collision model (known also as Discrete Element Method (DEM)) are validated with measurements obtained from a lab-scale spouted fluidized bed. The results confirm that the particle–grid method allows the variation of the fluid grid resolution independent of the particle size and consequently improves the calculation accuracy. In the second part of this work, the simulation results obtained from the extended Euler–Lagrange/DEM model are compared with the simulation results obtained from the Euler–Lagrange approach combined with a stochastic collision model. Two different fluidization mass flow rates are considered to analyse the ability of the used simulation approaches to predict the hydrodynamic behaviour of the gas spouted fluidized bed. The results show that both techniques can reproduce the right fluidization regimes including the bubble size and the bed expansion. Deviations from the experimental data in the jet zone and during the final stage of the bubble formation are, however, observed. The reasons of these discrepancies in predicting the dynamic behaviour of the bed, the advantages and limitations of the two approaches are demonstrated.

Al Rjoub et al. [10] proposed an analytical method is developed to study the dynamic behavior of functionally imperfect Euler-Bernoulli and Timoshenko graded beams with differing boundary conditions, namely, hinged-hinged, clamped-clamped, clamped-hinged, and clamped-free. A transfer matrix method is used to obtain the natural frequency equations. The modified rule of mixture is used to describe the material properties of the functionally graded beams having porosities. The porosities are assumed to be evenly distributed over the beam cross-section. In this study, the effects of boundary conditions, material volume fraction index, slenderness ratio, beam theory, and porosity on natural frequency are determined.

Yakubov et al. [11] proposed to different approaches of a pressure-velocity coupling method to account for density variations in cavitating two-phase flow simulations. Results obtained from two strategies are investigated in detail. A simpler engineering approach associated the variations of the local density solely with the changes of the vapor-volume fraction computed by a cavitation model and assumes incompressible vapor and water phases. A more elaborate method additionally accounts for the compressibility of the two individual fluid phases. Numerical issues of significance for engineering applications are discussed in the paper, such as the occurrence of ill-conditioned matrices or cavitation-model dependencies. The single-phase verification and validation study refers to prominent aerodynamic benchmarks, i.e. a convergent-divergent nozzle flow and the flow over a bump in a channel. Cavitating flow validations are concerned with a stationary flow over a hydrofoil. An unsteady cavitating flow over a NACA0015 hydrofoil is computed to demonstrate merits of the implemented compressible fluid method to simulate sheet and vapor cavitation including the collapse of a vapor cloud followed by a shock wave formation and propagation.

Zhang et al. [12] proposed on a modified Euler two-fluid model, simulations of a multiphase rotodynamic pump with two stages were carried out with medium combinations of air-water and air-crude. The characteristics of phase interaction and gas holdup were analyzed at different inlet gas void fractions (IGVFs), and inlet bubble diameters. The results show that the overall

changing trend of interphase forces is the same between the first and second stages at different IGVFs, but the magnitudes of interphase forces in the second stage are slightly smaller, especially for the medium combination of air-water. Moreover, the drag is more sensitive to the IGVF, while the lift and added mass force are more sensitive to the medium viscosity. As the increase of the inlet bubble diameter, the difference of the gas holdup effect in the pump increases gradually at IGVF = 9.0%, and the maximum almost occurs in the first stage guide vane (S1). When the bubble diameter increases to 0.7 mm, the degree of gas accumulation and gas-liquid velocity difference increase significantly, resulting in a significant increase of the disordered degree of lift and added mass force.

Guermond et al. [13] proposed new second-order method for approximating the compressible Euler equations is introduced. The method preserves all the known invariant domains of the Euler system: positivity of the density, positivity of the internal energy and the local minimum principle on the specific entropy. The technique combines a first-order, invariant domain preserving, Guaranteed Maximum Speed method using a Graph Viscosity (GMS-GV1) with an invariant domain violating, but entropy consistent, high-order method. Invariant domain preserving auxiliary states, naturally produced by the GMS-GV1 method, are used to define local bounds for the high-order method which is then made invariant domain preserving via a convex limiting process. Numerical tests confirm the second-order accuracy of the new GMS-GV2 method in the maximum norm, where 2 stands for second-order. The proposed convex limiting is generic and can be applied to other approximation techniques and other hyperbolic systems.

Li et al. [14] proposed high order well-balanced discontinuous Galerkin methods for the Euler equations with gravitation, which can preserve the discrete polytropic and isothermal hydrostatic balance states exactly. To achieve the well-balancedness, we propose to combine the numerical fluxes based on a generalized hydrostatic reconstruction, with an equilibrium state recovery technique and a novel source term approximation. Extensive one- and two-dimensional numerical examples are shown to demonstrate the performance of our well-balanced methods, and comparison with non well-balanced results is included to illustrate the importance of maintaining the balance between pressure gradient and gravitational force numerically.

Table 2.2: Numerical Methods for Solving IVPs Analysis

Author name and Reference	Techniques used	Merits	Demerits
Mao et al. [8]	Euler–Maruyama (EM)	Provides strong convergence theory for SDEs, specifically addresses strong-L^q convergence, and improves accuracy in numerical solutions for nonlinear SDEs under specific conditions.	Complexity in implementing and analyzing convergence under specific conditions, and the method may require careful tuning of parameters to ensure optimal performance.
Alobaid et al. [9]	particle–grid method in Euler–Lagrange/DEM simulations for fluidized beds	Improves calculation accuracy by varying fluid grid resolution independently of particle size, and both methods accurately predict fluidization regimes, bubble size, and bed expansion.	Discrepancies observed in the jet zone and bubble formation stages, and challenges in accurately predicting dynamic bed behavior and resolving discrepancies between simulation and experimental data.

Al Rjoub et al. [10]	transfer matrix method and modified rule of mixture	Provides a comprehensive analysis of natural frequency considering various factors like boundary conditions, porosity, and material properties, and uses a systematic approach for functionally graded beams.	Assumes evenly distributed porosity which may not represent all real-world scenarios, and complexity in handling different beam theories and boundary conditions could limit practical applicability.
Yakubov et al. [11]	pressure-velocity coupling methods for cavitating two-phase flow	Provides detailed analysis and validation for both single-phase and cavitating flows, demonstrating effective simulation of cavitation phenomena and shock wave formation using compressible fluid methods.	Potential numerical issues such as ill-conditioned matrices and cavitation-model dependencies, and increased complexity in simulations with compressible fluid methods.
Zhang et al. [12]	Simulated a multiphase rotodynamic pump using a modified Euler two-fluid model to analyze phase interactions, gas holdup, and the effects of inlet gas void fractions and bubble diameters.	Provides detailed insights into interphase forces and gas holdup effects, with observations of how drag, lift, and added mass forces vary with gas void fractions and bubble diameters.	Sensitivity to varying bubble diameters and inlet gas void fractions can lead to significant complexity in analyzing gas accumulation and force behavior, with potential for increased disordered force effects.
Guermond et al. [13]	Introduced a second-order method (GMS-GV2) for approximating compressible Euler equations, combining a first-order invariant domain preserving method with a high-order method via convex limiting.	Preserves critical invariant domains such as density positivity and entropy local minimum, achieves second-order accuracy, and applies convex limiting which is adaptable to other methods and hyperbolic systems.	Complexity of combining methods and applying convex limiting may increase implementation difficulty, and the method's performance may vary with different hyperbolic systems and approximation techniques.
Li et al. [14]	Developed high-order well-balanced discontinuous Galerkin methods for Euler equations with gravitation	Preserves discrete polytropic and isothermal hydrostatic balance exactly, improves numerical stability by maintaining balance between pressure gradient and gravitational force.	Complexity in implementation and potential computational cost of maintaining well-balanced states, with performance dependent on the effectiveness of the source term approximation and reconstruction techniques.

2.3 Taylor's Method

Taylor's Method is a numerical technique used for solving ordinary differential equations (ODEs) by approximating the solution using Taylor series expansions. It involves expanding the solution in terms of its derivatives and truncating the series after a specified number of terms to provide an approximation of the solution at each step. This method is highly accurate for smooth functions but can become computationally expensive as higher-order derivatives are required for better precision.

Malhotra et al. [15] proposed a boundary integral equation solver for computing Taylor relaxed states in non-axisymmetric solid and shell-like toroidal geometries. The computation of Taylor states in these geometries is a key element for the calculation of stepped pressure stellarator equilibria. The integral representation of the magnetic field in this work is based on the generalized

Debye source formulation, and results in a well-conditioned second-kind boundary integral equation. The

integral equation solver is based on a spectral discretization of the geometry and unknowns, and the computation of the associated weakly-singular integrals is performed with high-order quadrature based on a partition of unity. The resulting scheme for applying the integral operator is then coupled with an iterative solver and suitable preconditioners. Several numerical examples are provided to demonstrate the accuracy and efficiency of our method, and a direct comparison with the leading code in the field is reported.

Montavon et al. [16] proposed Deep Neural Networks (DNNs) are the gold standard for various challenging machine learning problems such as image recognition. Although these methods perform impressively well, they have a significant disadvantage, the lack of transparency, limiting the interpretability of the solution and thus the scope of application in practice. Especially DNNs act as black boxes due to their multilayer nonlinear structure. In this paper we introduce a novel methodology for interpreting generic multilayer neural networks by decomposing the network classification decision into contributions of its input elements. Although our focus is on image classification, the method is applicable to a broad set of input data, learning tasks and network architectures. Our method called deep Taylor decomposition efficiently utilizes the structure of the network by backpropagating the explanations from the output to the input layer. We evaluate the proposed method empirically on the MNIST and ILSVRC data sets.

Table 2.3: Taylor's Method Analysis

Author name and Reference	Techniques used	Merits	Demerits
Malhotra et al. [15]	Taylor relaxed states in non-axisymmetric toroidal geometries, using spectral discretization	Provides accurate and efficient computation of Taylor states in complex geometries, with a well-conditioned integral equation and effective numerical methods for high precision.	Complexity of implementing spectral discretization and high-order quadrature, potential computational expense of iterative solvers and preconditioners, and dependency on comparison with existing leading codes.
Montavon et al. [16]	Deep Taylor Decomposition for interpreting deep neural networks by decomposing classification decisions into contributions from input elements, using backpropagation of explanations from output to input.	Enhances transparency and interpretability of deep neural networks, applicable to various input data and network architectures, and evaluated on MNIST and ILSVRC datasets for empirical validation.	May add computational complexity due to the backpropagation of explanations and potential challenges in scaling to very large networks or complex tasks beyond image classification.

2.4 Simultaneous Fractional Differential Equations in IVPs

Simultaneous fractional differential equations (FDEs) in initial value problems (IVPs) represent an advanced and intricate field of study within mathematical analysis and applied sciences. These equations extend the classical differential equations by incorporating fractional derivatives, which are generalizations of integer-order derivatives. The inclusion of fractional orders allows for more accurate modeling of various phenomena, particularly those exhibiting memory and hereditary properties such as viscoelastic materials, anomalous diffusion processes, and biological systems.
In the context of IVPs, simultaneous FDEs involve multiple interdependent fractional differential equations that need to be solved concurrently, with specified initial conditions. The solutions to these systems provide valuable insights into the

dynamics of complex systems where traditional integer-order models fall short. Methods for solving these equations range from

analytical techniques, such as the Laplace transform and the Mittag-Leffler function, to numerical approaches like the finite element method, A domian decomposition method, and wavelet-based techniques.

One effective numerical method for solving simultaneous FDEs in IVPs is the Legendre wavelets method, which leverages the operational matrix of fractional derivatives to convert the problem into a system of algebraic equations. This transformation simplifies the solution process and enables efficient computation. The convergence and error analysis of the method ensure its correctness and feasibility, making it a powerful tool for addressing complex fractional systems. The study of simultaneous FDEs is essential for advancing our understanding of systems with fractional dynamics. By exploring these equations, researchers can develop more accurate models and solutions, thereby enhancing the application of fractional calculus in various scientific and engineering disciplines.

Hedayati et al. [17] proposed by using some fixed point technique such as Banach contraction principle and fixed point theorem of Krasnoselskii, we look into the positive solutions for fractional differential equation ${}^{c}D^{\alpha}u(t)$ equals to $f_1(t, u(t), {}^{c}D^{\beta 1}u(t), I^{\gamma 1}u(t))$ and $f_2(t, u(t), {}^{c}D^{\beta}{}_{2}u(t), I\gamma 2u(t))$ for each t belonging to $[0,t_0]$ and $[t_0, 1]$, respectively, with simultaneous Dirichlet boundary conditions, where cDα and Iα denote the Caputo fractional derivative and Riemann–Liouville fractional integral of order α, respectively. Some models are thrown to illustrate our results, too.

Malik et al. [18] determining a time dependent source term along with diffusion/temperature concentration from a non-local over-specified condition for a space-time fractional diffusion equation is considered. The space-time fractional diffusion equation involve Caputo fractional derivative in space and Hilfer fractional derivatives in time of different orders between 0 and 1. Under certain conditions on the given data we proved that the inverse problem is locally well-posed in the sense of Hadamard. Our method of proof based on eigenfunction expansion for which the eigenfunctions (which are Mittag-Leffler functions) of fractional order spectral problem and its adjoint problem are considered. Several properties of multinomial Mittag-Leffler functions are proved.

Tatar et al. [19] proposed the solution for the inverse problem is proved by using quasi-solution method which is based on minimizing an error functional between the output data and the additional data. In this context, an input–output mapping is defined and continuity of the mapping is established. The uniqueness of the solution for the inverse problem is also proved by using eigenfunction expansion of the solution and some basic properties of fractional Laplacian. A numerical method based on discretization of the minimization problem, steepest descent method and least squares approach is proposed for the solution of the inverse problem. The numerical method determines the exponents of the fractional time and space derivatives simultaneously. Numerical examples with noise-free and noisy data illustrate applicability and high accuracy of the proposed method.

Guerngar et al. [20] proposed the solution for the inverse problem is proved using the quasi-solution method which is based on minimizing an error functional between the output data and the additional data. In this context, an input-output mapping is defined and its continuity is established. The uniqueness of the solution for the inverse problem is proved by means of eigenfunction expansion of the solution of the forward problem and some basic properties of fractional Laplacian. A numerical method based on discretization of the minimization problem, namely the steepest descent method and a least squares approach, is proposed for the solution of the inverse problem. The numerical method determines the fractional exponents simultaneously. Finally, numerical examples with noise-free and noisy data illustrate applicability and high accuracy of the proposed method.

Jing et al. [21] proposed the uniqueness in identifying multiple parameters simultaneously in the one-dimensional time-fractional diffusion-wave equation of fractional time-derivative order $\in (0, 2)$ with the zero Robin boundary condition. Using the Laplace transform and a transformation formula, we prove the uniqueness in determining an order of the fractional derivative, a spatially varying potential, initial values and Robin coefficients simultaneously by boundary measurement data, provided that all the eigenmodes of an initial value do not vanish. Furthermore, for another formulation of inverse problem with input source term in place of initial value, by the uniqueness in the case of non-zero initial value and a Duhamel principle, we prove the simultaneous uniqueness in determining multiple parameters for a time-fractional diffusion-wave equation.

Table 2.4: Simultaneous Fractional Differential Equations

Author name and Reference	Techniques used	Merits	Demerits
Hedayati et al. [58]	Fixed point techniques, including the Banach contraction principle and Krasnoselskii's fixed point theorem, to analyze positive solutions for fractional differential equations with Caputo derivatives and Riemann–Liouville integrals.	Provides a robust framework for establishing the existence of solutions under Dirichlet boundary conditions, applicable to various models.	The reliance on fixed point methods may limit the applicability to more complex or nonlinear scenarios.
Malik et al. [59]	Eigenfunction expansion using Mittag-Leffler functions to analyze a space-time fractional diffusion equation with Caputo and Hilfer fractional derivatives, proving local well-posedness.	Establishes a strong foundation for solving inverse problems and demonstrates the applicability of Mittag-Leffler functions in fractional calculus.	The complexity of the eigenfunction approach may limit straightforward implementation in practical scenarios.
Tatar et al. [60]	Quasi-solution method to minimize error functionals, eigenfunction expansion for uniqueness proof, and numerical methods involving discretization, steepest descent, and least squares for solving inverse problems.	Effectively determines fractional derivative exponents simultaneously while ensuring high accuracy even with noisy data.	The method may require careful tuning of parameters and can be sensitive to noise in the data.
Guerngar et al. [61]	Quasi-solution method for minimizing error functionals,	Achieves simultaneous determination of fractional exponents with high	Sensitivity to noise and the need for careful parameter tuning

	eigenfunction expansion for uniqueness proof, and numerical methods including steepest descent and least squares for solving the inverse problem.	accuracy, demonstrating effectiveness with both noise-free and noisy data.	may complicate practical applications.
Jing et al. [62]	Laplace transform and transformation formula to establish uniqueness in identifying multiple parameters for time-fractional diffusion-wave equations under zero Robin boundary conditions.	Effectively demonstrates simultaneous identification of fractional derivative orders, potential, initial values, and Robin coefficients from boundary measurements.	Dependence on non-vanishing eigenmodes may limit applicability in certain scenarios with insufficient boundary data.

3. Summary

This collection focuses on Initial Value Problems (IVPs) and various numerical methods for their solutions, including Modified Euler's Method, Taylor's Method, and Modified Fractional Methods. It emphasizes the role of Special Functions in differential methods, particularly the Mittag-Leffler Function, in addressing IVPs. Additionally, it covers definitions and applications of Fractional Calculus in IVPs, exploring both single and simultaneous Fractional Differential Equations to highlight their significance in modeling complex dynamic systems.

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